University of Rochester

PHY235 Term Paper

Understanding Precession

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1 Introduction

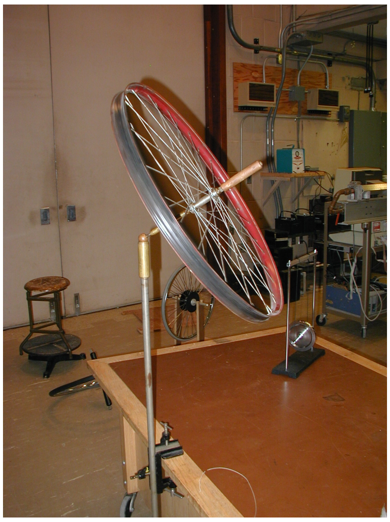


Figure 1: Bicycle wheel gyroscope demonstration used at the University of Rochester. Image credit: Thang V. Nguyen[6]

The mysterious ability of a spinning top to balance at seemingly impossible angles has fascinated observers since ancient time. Today, physics students are usually first introduced to this phenomenon by studying a specific type of top: the gyroscope. When the tip of a stationary gyroscope is placed on a pivot and released, it falls to the side under the force of gravity. However, when a spinning gyroscope is placed in the same orientation, it does not fall over. Instead, the gyroscope tips slightly down and begins to rotate around the pivot. This rotation is called precession. Upon looking closer, a careful observer will notice that as the gyroscope precesses its tip also rotates up and down. This motion is known as nutation.

This precession can be explained relatively easily using simple Newtonian mechanics as a result of the conservation of angular momentum. However, in addition to neglecting to explain the origin of the tops nutation, this mathematical explanation doesn't make it any easier to understand how the gyroscope defies gravity[1]. This leaves many first year physics students (including the author) confounded by what still appears to be an intuitively impossible result.

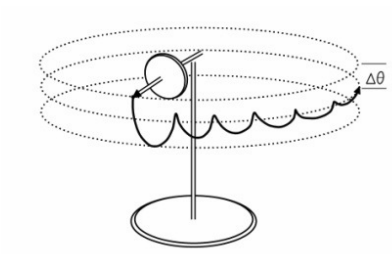
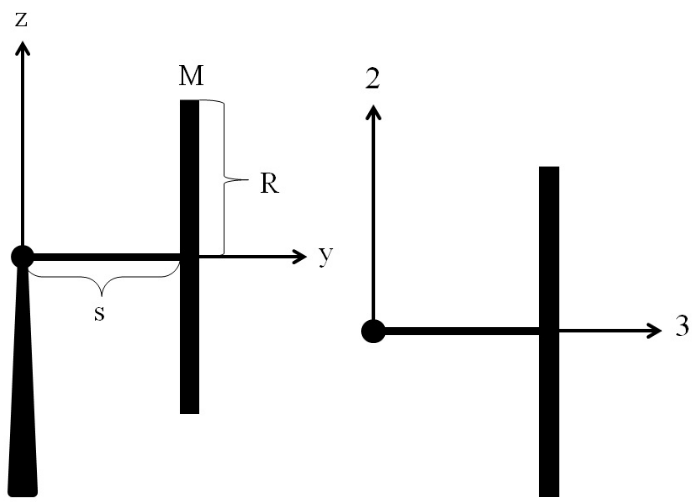


Figure 2: A precessing and nutating gyroscope. Image credit: Svilen Kostov and Daniel Hammer [5]

Luckily, the powerful Lagrangian formulation of classical mechanics provides the tools for a more explanatory analysis of this phenomenon. These techniques allow us to find equations of motion for the gyroscope and, with the aid of computer simulations, gain a better intuitive understanding of how the gyroscope accomplishes its seemingly impossible balancing act.

[1]Richard Feynman describes this explanation as "a miracle involving right angles and circles, and twists and right-hand screws" (Feynman Lectures 20-6)[4, pg. 20-6].

2 Describing the Bicycle Wheel Gyroscope



(a) Lab frame (x,y,z)
(b) Body frame (1,2,3)

Figure 3: Setup of the experiment with labeled dimensions and axes in both the lab (fixed) and body (rotating) reference frames

The gyroscope is often demonstrated in introductory physics classes using a bicycle wheel mounted on a pivot, so this is the system we will analyze. The bicycle wheel is attached via a pivot to a pedestal that supports one end at a fixed height. The wheel is then set spinning and released with the axle horizontal (Figure 3a).

To describe this motion, we must first pick appropriate coordinates in both the laboratory (x,y,z) and body fixed (1,2,3) frames. In the lab frame, let the vertical be the z axis and chose the initial orientation of the wheel's axle before it as released as the y axis (Figure 3a). In keeping with conventions, let the body fixed 3 axis point along the axle of the wheel (Figure 3b).

The physical geometry of the bicycle wheel gyroscope can be described with an inertia tensor. The inertia tensor of an actual bicycle wheel with all its spokes and details would be immensely complicated to calculate, so we will model the bicycle wheel as a ring of radius R and mass M. Let the distance from the pivot to the center of the wheel be s. The entire system rotates around the end of the axle, so we will need to find the inertia tensor about this point. To find this inertia tensor, we will consider the moment of the ring about its center and then use the parallel axis theorem to find the moment of inertia about the tip of the axle. The moment of inertia of the ring about its center can be easily computed:

$$I_{ring} = \begin{bmatrix} \frac{MR^2}{2} & 0 & 0 \\ 0 & \frac{MR^2}{2} & 0 \\ 0 & 0 & MR^2 \end{bmatrix}$$

Note that this inertia tensor obeys the perpendicular axis theorem for thin planes: $I_1 = I_2 = \frac{I_3}{2}$. However, the gyroscope does not rotate about the center of the ring but rather about the end of it's axle. The parallel axis theorem allows us to easily shift the moment of inertia by a displacement $< -s, 0, 0 >$.

$$I_{gyroscope} = \begin{bmatrix} M(\frac{R^2}{2} + s^2) & 0 & 0 \\ 0 & M(\frac{R^2}{2} + s^2) & 0 \\ 0 & 0 & MR^2 \end{bmatrix}$$

As we expect, this inertia tensor is already diagonalized because the 3-axis is a principle axis of the ring.

For the purpose of numerical calculations, we will assign values to each of these parameters. A typical bicycle wheel has a radius of approximately 0.3m and weighs around 13lbs (6 kg). We will consider a short 0.02m axle that exaggerates the motion we are examining.

3 Solving the Bicycle Wheel Gyroscope with Lagrangian Mechanics

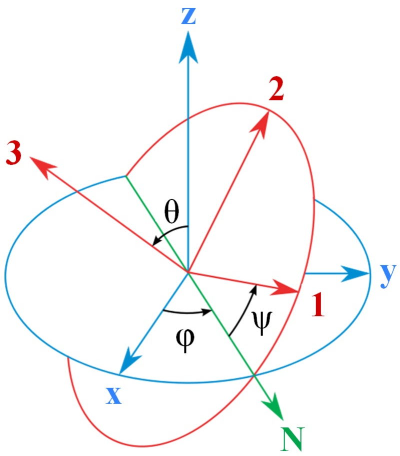


Figure 4: Euler angles. Image Credit: Modified from Lionel Brits [1]

In order to apply Lagrangian mechanics to the problem of the gyroscope it is easiest to express the gyroscope's position in terms of Euler angles. If we chose the conventions shown in figure 4, then the precessional velocity of the gyroscope will be $\dot{\phi}$, the nutation velocity will be $\dot{\theta}$, and the angular velocity of the spinning wheel itself will be $\dot{\psi}$.

In order to write the Lagrangian, we must first find expressions for the kinetic and potential energy of the gyroscope. The kinetic energy can be found by projecting the angular velocities of the gyroscope onto the body fixed axes:

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta}\cos\psi \tag{1}$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta}\sin\psi \tag{2}$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{3}$$

Since the $I_1 = I_2$ as calculated on the previous page, we can then easily write out the kinetic energy:

$$\begin{aligned} T &= \frac{1}{2} I_1\omega_1^2 + \frac{1}{2} I_2\omega_2^2 + \frac{1}{2} I_3\,\omega_3^2 & (4) \\ &= \frac{1}{2} I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 & (5) \end{aligned}$$

The potential energy is simply gravity:

$$U = mgs\cos\theta \tag{6}$$

Finally the total Lagrangian is just $T - U$:

$$L = \frac{1}{2} I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - mgs\cos\theta \tag{7}$$

The normal process of applying Lagrange's equation of motion to this Lagrangian by hand would be incredibly tedious. However, we can immediately see from equation 7 that both ϕ and ψ are cyclic variables:

$$\dot{p_\phi} = \frac{\partial L}{\partial \phi} = 0 \tag{8}$$

$$\dot{p_\psi} = \frac{\partial L}{\partial \psi} = 0 \tag{9}$$

Which implies that both angular momentums are constant:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta = \text{constant} \tag{10}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = \text{constant} \tag{11}$$

By combining these two constants, we can thus derive an expression for the precession velocity [8]:

$$\dot{\phi} = \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta} \tag{12}$$

However, at this point we reach an impasse. Fully solving these equations of motion by hand would be very difficult and extremely messy. Instead, we will use Mathematica's differential equation solver to numerically solve Lagrange's equations of motion [7]. These calculations yield the following results for our gyroscope:

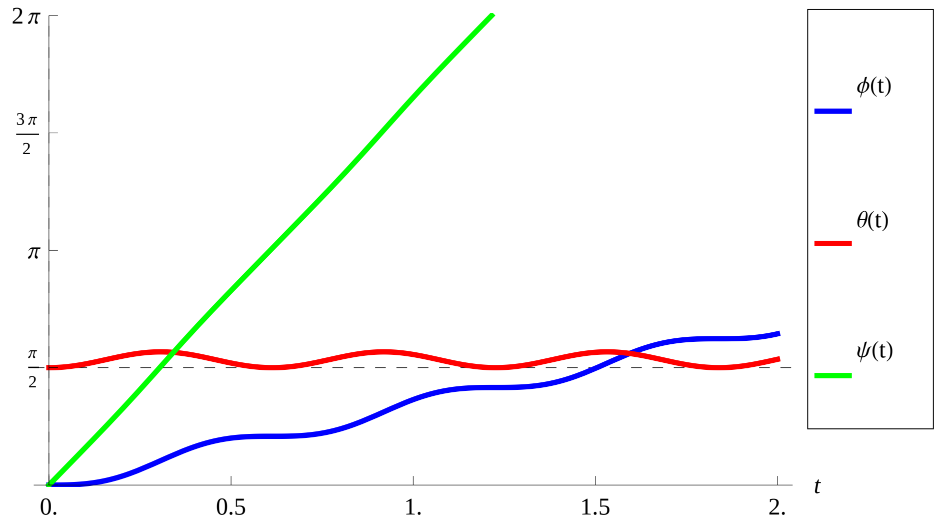


Figure 5: Solutions to Lagrange's Equations of Motion

The solutions in Figure 5 hint at the behavior of the gyroscope: ψ is a line (the angular velocity of the wheel remains constant) and θ oscillates back and forth above $\frac{\pi}{2}$. ϕ, the precession angle, increases, but its velocity is not constant: as the gyroscope nutates back and forth, ϕ (which, as seen in equation 12, is dependent on θ), $\dot{\phi}$ changes.

4 Explaining Precession

We are now in a position to illustrate an intuitive explanation of precession. Imagine that the bicycle wheel gyroscope is set spinning and held horizontally ($\theta = \frac{\pi}{2}$). The angular momentum of the wheel is along the body-fixed 3 axis and thus can be attributed entirely to the rotation of the wheel about that axis.

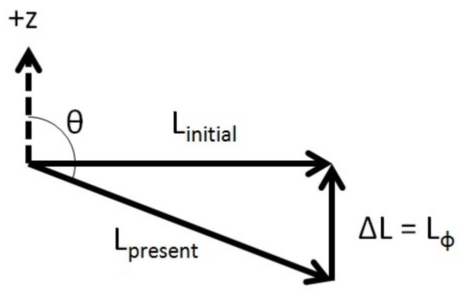


Figure 6: An angular momentum component in the z direction emerges as the gyroscope tips downward

When we let go of the wheel, it begins to fall under the force of gravity. However, while the angular momentum of the wheel *about the body-fixed 3 axis* is now pointing down below $\frac{\pi}{2}$ [2] (figure 6). The conservation of angular momentum requires that the total angular momentum remain constant. Thus, a *new* component of angular momentum (L_ϕ in figure 6) must emerge so that the vector sum of $L_{present}$ and L_ϕ remains constant. This new component of angular momentum creates a rotation about the z-axis. However, rotation about the z axis is exactly the phenomenon of precession! This new component of angular momentum is the angular momentum of the precessional motion.

Since angular momentum is directly proportional to angular velocity, it is now obvious that the precessional velocity $\dot{\phi}$ (equation 12) must depend on θ. This relationship can be illustrated clearly by graphing $\dot{\phi}$ and θ verses time (figure 7). As the gyroscope nutates, its precessional velocity changes to conserve the total angular momentum.

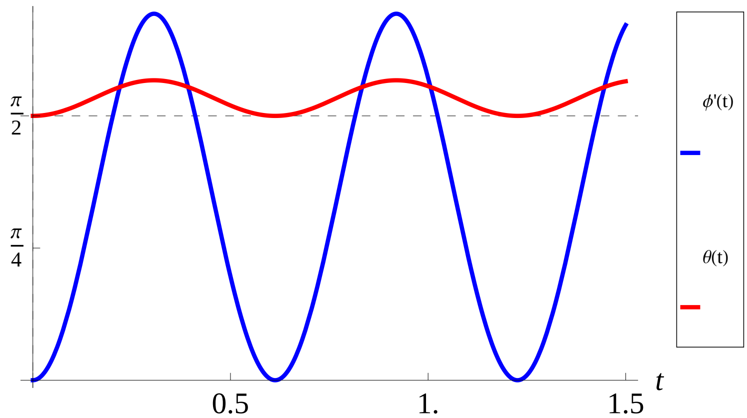


Figure 7: An angular momentum component in the z direction emerges as the gyroscope tips downward

This explanation of the motion shown in figure 7 makes sense until the direction of the nutation velocity changes. However, when the wheel suddenly begins to move upward, against the force

[2] Remember that θ is measured from the positive z axis, so the larger θ gets, the farther the gyroscope dips below the horizontal.

of gravity, our argument based on the conservation of angular momentum seems at a loss. This motion can be explained by looking more closely at the path followed by an individual point on the wheel.

To follow a single point on the wheel throughout its rotation we will choose a point on the wheel defined by a position vector in the body fixed frame and then transform that point into the laboratory frame using the fact that:

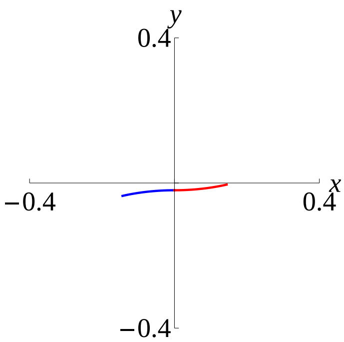
$$< x, y, z >= \lambda^{-1} \cdot < 1, 2, 3 > \tag{13}$$

Where λ is the Euler angle rotation matrix from the space fixed frame to the body fixed frame. Using Mathematica, we can then compute a time dependent rotation matrix λ^{-1} that will transform between the body fixed and lab reference frames [3]:

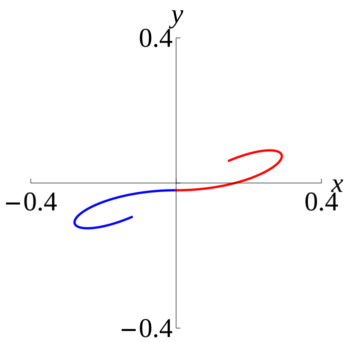
$$\lambda^{-1} = \begin{bmatrix} \cos\phi(t)\cos\psi(t) - \sin\phi(t)\cos\theta(t)\sin\psi(t) & -\cos\phi(t)\sin\psi(t) - \sin\phi(t)\cos\theta(t)\cos\psi(t) & \sin\phi(t)\sin\theta(t) \\ \sin\phi(t)\cos\psi(t) + \cos\phi(t)\cos\theta(t)\sin\psi(t) & -\sin\phi(t)\sin\psi(t) + \cos\phi(t)\cos\theta(t)\cos\psi(t) & -\cos\phi(t)\sin\theta(t) \\ \sin\theta(t)\sin\psi(t) & \sin\theta(t)\cos\psi(t) & \cos\theta(t) \end{bmatrix}$$

We will chose a point on the whee defined by the position vector $< 0, R, s >$. Plugging λ^{-1} into equation 13 gives a vector function $P(t)$ that describes the motion of a point beginning at the top of the wheel. Applying the same transformation matrix to the position vector $< 0, -R, s >$ gives us a similar vector function $Q(t)$ that describes a point beginning at the bottom of the wheel.

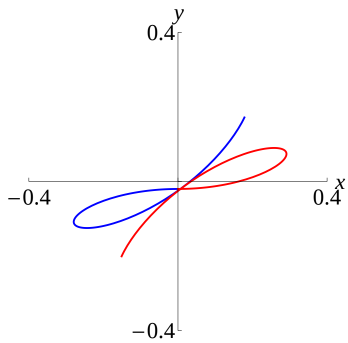
Graphing both $P(t)$ (blue) and $Q(t)$ (red) in the x-y plane reveals an interesting motion:



(a) t = 0.1 s. P going down, Q going up



(b) t = 0.5 s. P at the bottom, Q at the top



(c) t = 0.8 s. P going up, Q going down

Figure 8: Trajectories of two points on the wheel. Point P (blue) begins on the top of the wheel, while point Q (red) starts at the bottom.

Both points start together in the x-y plane and move apart along nearly straight lines (figure 8a). However, as point P begins to rotate around towards the bottom of the wheel, it is pushed out. Likewise, as point Q rotates up towards the top of the wheel, it is pulled in (figure 8b). This motion continues as the wheel comes around again: as P comes back around to the top of the wheel again in figure 8c it is pulled in, while point Q is pushed out.

The motion of precession is providing a fictitious force that changes the direction of motion of these points.These forces are opposite in direction and thus do not apply a force to the center of mass. However, they *do* create a torque about the center of the wheel (figure 9), opposing the force of gravity! It is this fictitious force that keeps the gyroscope upright.

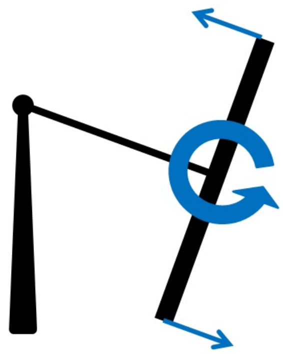


Figure 9: The forces required to alter the trajectories of the points on the wheel exert a torque about the axis, supporting it against the torque of gravity

5 Friction and Stable Precession

While precession is immediately visible when looking at an actual bicycle wheel gyroscope, nutation is much more difficult to observe. This can easily be explained by considering the frictional forces acting at the pivot of the wheel. It is reasonable to assume these forces are approximately constant, changing sign to oppose the movement of the gyroscope. However, since $\dot{\theta}$ is generally much greater than $\dot{\phi}$, the nutation of the gyroscope is slowed much more rapidly by frictional forces than the precession. Over a short time the nutation motion will dampen out, leaving the gyroscope to stably precess with a fixed $\theta > \pi/2$.

In order to quantitatively investigate this progression we can include a generalized constant frictional force to Lagrange's equations of motion. Frictional forces are oriented opposite to the velocity of the particle, so we will model each force as an arbitrary constant multiplied by the velocity unit vector in each direction:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = -F\hat{\dot{\phi}} = -F\frac{\dot{\phi}}{|\dot{\phi}|} \tag{14}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = -F\hat{\dot{\theta}} = -F\frac{\dot{\theta}}{|\dot{\theta}|} \tag{15}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} = -F\hat{\dot{\psi}} = -F\frac{\dot{\psi}}{|\dot{\psi}|} \tag{16}$$

Where L is still given by equation 7. Again we can solve this system numerically using Mathematica. The decay to stable precession is most easily observed by considering a graph of $\theta(t)$ vs. $\phi(t)$. When $F = 0$ we observe that this curve takes the shape of a cycloid:

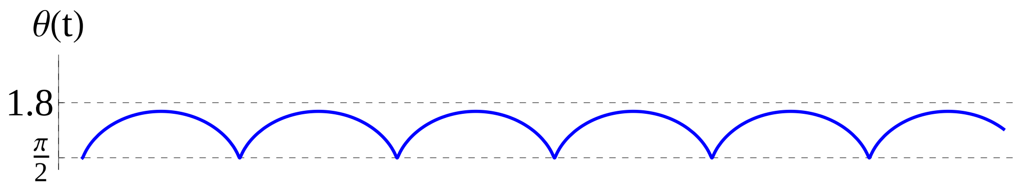


Figure 10: The point of the gyroscope traces out a cycloid as it precesses

However, when we introduce a frictional force, this cycloid pattern decays:

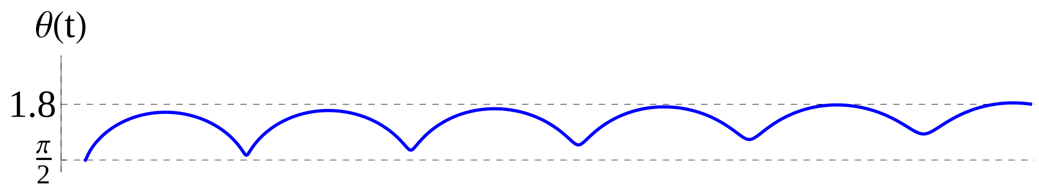


Figure 11: F= -0.5; t= 0 to t $\approx$ 3.5

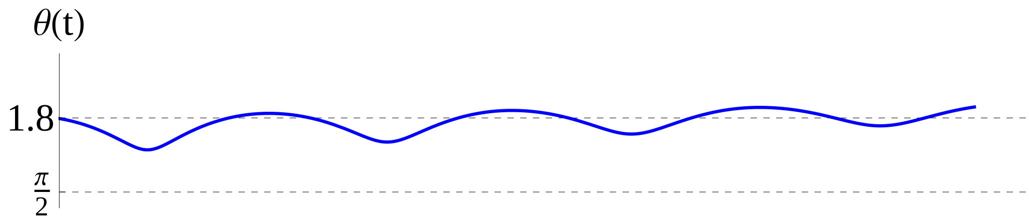


Figure 12: F= -0.5; t $\approx$ 3.5 to t $\approx$ 6.5

The gyroscope eventually settles at a constant θ. Since it continues to precess around we should expect (by our arguments in the previous section) that it settles at an angle $\theta > \pi/2$, which is indeed the case (here $\theta \approx 1.8$).

6 Conclusion

Despite its complexity, the bicycle wheel gyroscope is a rightfully iconic physics demonstration. However, to avoid confusion, it is important that some effort be made to make gyroscopic motion intuitively acceptable to students. Many intuitively satisfying models of this motion exist [2][4][5]. The use of tools such as Lagrangian mechanics and numerical differential equation analysis allows us to more effectively communicate these explanations.

References

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